

Cosmological General Relativity With Scale Factor and Dark Energy

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Abstract In this paper the four-dimensional (4-D) space-velocity Cosmological General Relativity of Carmeli is developed by a general solution of the Einstein field equations. The Tolman metric is applied in the form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \tau^2 dv^2 - e^\mu dr^2 - R^2 (d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

where $g_{\mu\nu}$ is the metric tensor. We use comoving coordinates $x^\alpha = (x^0, x^1, x^2, x^3) = (\tau v, r, \theta, \phi)$, where τ is the Hubble-Carmeli time constant, v is the universe expansion velocity and r, θ and ϕ are the spatial coordinates. We assume that μ and R are each functions of the coordinates τv and r .

The vacuum mass density ρ_Λ is defined in terms of a cosmological constant Λ ,

$$\rho_\Lambda \equiv -\frac{\Lambda}{\kappa\tau^2}, \quad (2)$$

where the Carmeli gravitational coupling constant $\kappa = 8\pi G/c^2\tau^2$, where c is the speed of light in vacuum. This allows the definitions of the effective mass density

$$\rho_{eff} \equiv \rho + \rho_\Lambda \quad (3)$$

and effective pressure

$$p_{eff} \equiv p - c\tau\rho_\Lambda, \quad (4)$$

where ρ is the mass density and p is the pressure. Then the energy-momentum tensor

$$T_{\mu\nu} = \tau^2 \left[\left(\rho_{eff} + \frac{p_{eff}}{c\tau} \right) u_\mu u_\nu - \frac{p_{eff}}{c\tau} g_{\mu\nu} \right], \quad (5)$$

where $u_\mu = (1, 0, 0, 0)$ is the 4-velocity. The Einstein field equations are taken in the form

$$R_{\mu\nu} = \kappa \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right), \quad (6)$$

where $R_{\mu\nu}$ is the Ricci tensor, $\kappa = 8\pi G/c^2\tau^2$ is Carmeli's gravitation constant, where G is Newton's constant and the trace $T = g^{\alpha\beta} T_{\alpha\beta}$. By solving the field

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equations (6) a space-velocity cosmology is obtained analogous to the Friedmann-Lemaître-Robertson-Walker space-time cosmology.

We choose an equation of state such that

$$p = w_e c \tau \rho, \quad (7)$$

with an evolving state parameter

$$w_e(R_v) = w_0 + (1 - R_v) w_a, \quad (8)$$

where $R_v = R_v(v)$ is the scale factor and w_0 and w_a are constants.

Carmeli's 4-D space-velocity cosmology is derived as a special case.

Keywords cosmology theory; space-velocity; cosmological constant; dark energy; scale factor.

1 Introduction

Cosmological General Relativity (CGR) is a 5-D time-space-velocity theory[1, 2] of the cosmos, for one dimension of cosmic time, three of space and one of the universe expansion velocity. Cosmic time is taken to increase from the present epoch $t = 0$ toward the big bang time $t = \tau$, where τ is the Hubble-Carmeli time constant. The expansion velocity $v = 0$ at the present epoch and increases toward the big bang. In this paper the cosmic time t is held fixed ($dt = 0$) and measurements are referred to the present epoch of cosmic time. This is a reasonable approach since the time duration over which observations are made is negligible compared to the travel time of the emitted light from the distant galaxy.

Hence, in this paper we examine the four-dimensional (4-D) space-velocity of CGR. A general solution to the Einstein field equations in the space-velocity domain is obtained, analogous to the Friedmann-Lemaître-Robertson-Walker (FLRW) solution of space-time cosmology. The main emphasis herein is to develop a cosmology having a scale factor R_v dependent on the expansion velocity which in turn can be expressed in terms of the cosmological redshift z . This will enable a set of well defined tools for the analysis of observational data where the cosmological redshift plays a central role. The resulting cosmology is used to model a small set of SNe Ia data.

We will derive Carmeli's cosmology as a special case where the scale factor is held fixed. Two principal results of Carmeli's cosmology is the prediction of the accelerated expansion of the universe[3] and the description of spiral galaxy rotation curves without additional dark matter[4]. We continue to support those results within this paper with a theoretical framework that accommodates a more varied parameter space.

For our purposes, a vacuum mass density ρ_Λ is defined in terms of a cosmological constant Λ by

$$\rho_\Lambda \equiv -\frac{\Lambda}{\kappa \tau^2}, \quad (9)$$

where the Carmeli gravitational coupling constant $\kappa = 8\pi G/c^2 \tau^2$, where c is the speed of light in vacuum and τ is the Hubble-Carmeli time constant. In a previous article[5] we hypothesized that the observable universe is one of two

black holes joined at their event horizons. From this perspective we show that the vacuum density of the observable universe and the universe black hole entropy have the relation $\rho_A \propto S^{-1}$, where S is the Bekenstein-Hawking entropy[6, 7] of the black hole. We also will use an evolving two parameter equation of state $w_e(R_v)$, dependent on the scale factor R_v , which allows for the evolution of the effect of dark energy on the pressure[8].

2 The Metric

Assuming the matter in the universe to be isotropically distributed we will adopt a metric that is spatially spherical symmetric. Furthermore, the spatial coordinates will be co-moving such that galaxies expanding along the same geodesic curve are motionless with respect to one another. In this manner we can compare observations between galaxies moving along different geodesic paths.

A general derivation of the metric we will use was given by Tolman[9] and is taken in the simplified form defined by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \tau^2 dv^2 - e^\mu dr^2 - R^2 (d\theta^2 + \sin^2\theta d\phi^2), \quad (10)$$

where $g_{\mu\nu}$ is the metric tensor. The comoving coordinates are $x^\alpha = (x^0, x^1, x^2, x^3) = (\tau v, r, \theta, \phi)$, where τ is the Hubble-Carmeli time constant, v is the universe expansion velocity and r, θ and ϕ are the spatial coordinates. Assume that the functions μ and R are functions of coordinates τv and r .

The constant τ is related to the Hubble constant H_0 at zero distance and zero gravity by the relation $h = 1/\tau$ where measurements of H_0 at very close (local) distances are used to determine the value of h . At this writing the accepted value[10] is

$$h = 72.17 \pm 0.84 \pm 1.64 \text{ km/s/Mpc}. \quad (11)$$

Therefore

$$\tau = (4.28 \pm 0.15) \times 10^{17} \text{ s} = 13.56 \pm 0.48 \text{ Gyr}. \quad (12)$$

From (10) the non-zero components of the metric tensor $g_{\mu\nu}$ are given by

$$g_{00} = 1, \quad (13)$$

$$g_{11} = -e^\mu, \quad (14)$$

$$g_{22} = -R^2, \quad (15)$$

$$g_{33} = -R^2 \sin^2\theta. \quad (16)$$

The choice of the particular metric (10) determines that the 4-velocity of a point moving along the geodesic curve is given by

$$u_\mu = u^\mu = dx^\mu/ds = (1, 0, 0, 0). \quad (17)$$

The universe expands by the null condition $ds = 0$. For a spherically symmetric expansion one has $d\theta = d\phi = 0$. The metric (10) then gives

$$\tau^2 dv^2 - e^\mu dr^2 = 0, \quad (18)$$

which yields

$$\frac{dr}{dv} = \tau e^{-\mu/2}. \quad (19)$$

To determine the functions μ and R we need to solve the Einstein field equations.

3 The Field Equations

The Einstein field equations with a cosmological constant Λ term are taken in the form

$$R_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa \left(T'_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T' \right), \quad (20)$$

where $R_{\mu\nu}$ is the Ricci tensor, $T'_{\mu\nu}$ is the energy-momentum tensor, $T' = g^{\alpha\beta} T'_{\alpha\beta}$ is its trace and κ is Carmeli's gravitational coupling constant given by

$$\kappa = \frac{8\pi G}{c^2 \tau^2}, \quad (21)$$

where G is Newton's gravitational constant and c is the speed of light in vacuum. If we add the tensor $\Lambda g_{\mu\nu}$ to the Ricci tensor $R_{\mu\nu}$, the covariant derivative of the new tensor is still zero. That is

$$\nabla_\nu (R_{\mu\nu} + \Lambda g_{\mu\nu}) = \nabla_\nu R_{\mu\nu} + \Lambda \nabla_\nu g_{\mu\nu} = 0, \quad (22)$$

since the covariant derivatives of the Ricci tensor and the metric tensor are both zero.

We will move the cosmological constant term from the left hand side (l.h.s.) to the right hand side (r.h.s.) of (20) to make it a component of the energy-momentum tensor giving

$$R_{\mu\nu} = \kappa \left(T'_{\mu\nu} - \frac{\Lambda}{\kappa} g_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T' \right). \quad (23)$$

Since the covariant derivative of the energy-momentum tensor $\nabla_\nu (T'_{\mu\nu}) = 0$, the covariant derivative of the r.h.s. of (23) equals zero. The Λ term is absorbed into a new energy-momentum tensor by the form

$$T_{\mu\nu} = \tau^2 \left[\left(\rho + \frac{p}{c\tau} \right) u_\mu u_\nu - \left(\frac{p}{c\tau} + \frac{\Lambda}{\tau^2 \kappa} \right) g_{\mu\nu} \right], \quad (24)$$

where ρ is the mass density, p is the pressure and $u_\mu = u^\mu = dx^\mu/ds = (1, 0, 0, 0)$ is the 4-velocity. The τ^2 factor on the r.h.s. of (24) is an artifact of convenience in solving the field equations.

Taking the trace T of the new energy-momentum tensor yields

$$\begin{aligned} T &= g^{\alpha\beta} T_{\alpha\beta} = \tau^2 \left[\rho + \frac{p}{c\tau} - 4 \left(\frac{p}{c\tau} + \frac{\Lambda}{\tau^2 \kappa} \right) \right] \\ &= \tau^2 \rho - 3 \frac{\tau}{c} p - 4 \tau^2 \left(\frac{\Lambda}{\tau^2 \kappa} \right). \end{aligned} \quad (25)$$

Define the vacuum mass density ρ_Λ in terms of the cosmological constant Λ ,

$$\rho_\Lambda \equiv -\frac{\Lambda}{\kappa\tau^2}. \quad (26)$$

By this definition we are defining ρ_Λ to be a negative density, unless Λ is negative. This is contrary to the design of the standard model but is in keeping with the Behar-Carmeli cosmological model[2].

Then in terms of the vacuum mass density ρ_Λ , define the effective mass density and effective pressure,

$$\rho_{eff} \equiv \rho + \rho_\Lambda, \quad (27)$$

and

$$p_{eff} \equiv p - c\tau\rho_\Lambda. \quad (28)$$

With the definitions for the effective mass density and the effective pressure the energy-momentum tensor (24) becomes

$$T_{\mu\nu} = \tau^2 \left[\left(\rho_{eff} + \frac{p_{eff}}{c\tau} \right) u_\mu u_\nu - \frac{p_{eff}}{c\tau} g_{\mu\nu} \right]. \quad (29)$$

In terms of the effective mass density and pressure the only non-zero components of $T_{\mu\nu}$ are given by

$$T_{00} = \tau^2 \left(\rho_{eff} + \frac{p_{eff}}{c\tau} \right) - \tau^2 \left(\frac{p_{eff}}{c\tau} \right) = \tau^2 \rho_{eff}, \quad (30)$$

$$T_{11} = \frac{\tau}{c} e^\mu p_{eff}, \quad (31)$$

$$T_{22} = \frac{\tau}{c} R^2 p_{eff}, \quad (32)$$

$$T_{33} = \frac{\tau}{c} R^2 \sin^2 \theta p_{eff}. \quad (33)$$

The trace of $T_{\mu\nu}$ in terms of the effective mass density and pressure is

$$T = \tau^2 \rho_{eff} - 3 \frac{\tau}{c} p_{eff}. \quad (34)$$

Substituting for the defined values of ρ_{eff} and p_{eff} , the trace (34) expands out to

$$T = \tau^2 \rho - 3 \frac{\tau}{c} p - 4\tau^2 \left(\frac{\Lambda}{\tau^2 \kappa} \right), \quad (35)$$

which is equal to (25). With our definition for the new energy-momentum tensor the field equations now take the form

$$R_{\mu\nu} = \kappa \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right). \quad (36)$$

Here we write out the nonvanishing components of the Ricci tensor. A dot ($\dot{}$) denotes partial differentiation with respect to $x^0 = \tau v$ and a prime (\prime) denotes partial differentiation with respect to $x^1 = r$. This follows [1], (Sect. 4.3.)

$$R_{00} = -\frac{1}{2}\ddot{\mu} - \frac{2}{R}\ddot{R} - \frac{1}{4}\dot{\mu}^2, \quad (37)$$

$$R_{01} = \frac{1}{R}R'\dot{\mu} - \frac{2}{R}\dot{R}', \quad (38)$$

$$R_{11} = e^\mu \left(\frac{1}{2}\ddot{\mu} + \frac{1}{4}\dot{\mu}^2 + \frac{1}{R}\dot{\mu}\dot{R} \right) + \frac{1}{R}(\mu'R' - 2R''), \quad (39)$$

$$R_{22} = R\ddot{R} + \frac{1}{2}R\dot{R}\dot{\mu} + \dot{R}^2 + 1 - e^{-\mu} \left(RR'' - \frac{1}{2}RR'\mu' + R'^2 \right), \quad (40)$$

$$R_{33} = \sin^2\theta R_{22}. \quad (41)$$

Expanding the r.h.s. of the field equations (36) yields

$$R_{00} = \frac{1}{2}\tau^2\rho_{eff} + \frac{3}{2}\frac{\tau}{c}p_{eff}, \quad (42)$$

$$R_{01} = 0, \quad (43)$$

$$R_{11} = \frac{1}{2}\tau^2 e^\mu \rho_{eff} - \frac{1}{2}\frac{\tau}{c}e^\mu p_{eff}, \quad (44)$$

$$R_{22} = \frac{1}{2}\tau^2 R^2 \rho_{eff} - \frac{1}{2}\frac{\tau}{c}R^2 p_{eff}, \quad (45)$$

$$R_{33} = \frac{1}{2}\tau^2 R^2 \sin^2\theta \rho_{eff} - \frac{1}{2}\frac{\tau}{c}R^2 \sin^2\theta p_{eff}. \quad (46)$$

We obtain our first independent field equation by multiplying (44) by $e^{-\mu}$ and adding the result to (42). This operation will eliminate the $\ddot{\nu}$ and $\dot{\mu}^2$ terms leaving

$$-\frac{2}{R}\ddot{R} + \frac{1}{R}\dot{\mu}\dot{R} + e^{-\mu}\frac{1}{R}(\mu'R' - 2R'') = \kappa\tau^2\rho_{eff} + \kappa\frac{\tau}{c}p_{eff}. \quad (47)$$

By multiplying (45) by $2/R^2$ and adding the result to (47), the \ddot{R} and p_{eff} terms will be eliminated leaving

$$\frac{\dot{\mu}\dot{R}}{R} + \left(\frac{\dot{R}}{R}\right)^2 + \frac{1}{R^2} + e^{-\mu} \left[\frac{\mu'R'}{R} - \frac{2R''}{R} - \left(\frac{R'}{R}\right)^2 \right] = \kappa\tau^2\rho_{eff}. \quad (48)$$

The next basic independent field equation is obtained by substituting for the value of the expression $e^{-\mu}(\mu'R' - 2R'')/R$ in (48). To do that, multiply (45) by $2/R^2$ and move all other terms to the r.h.s. leaving

$$\begin{aligned} e^{-\mu} \left(\frac{\mu'R'}{R} - \frac{2R''}{R} \right) = & -\frac{2\ddot{R}}{R} - \frac{\dot{R}\dot{\mu}}{R} - 2\left(\frac{\dot{R}}{R}\right)^2 - \frac{2}{R^2} \\ & + 2e^{-\mu} \left(\frac{R'}{R} \right)^2 + \kappa\tau^2\rho_{eff} - \kappa\frac{\tau}{c}p_{eff}. \end{aligned} \quad (49)$$

Substitute the expression in (49) for its corresponding expression in (48). After eliminating some terms, combining other terms, multiplying both sides by $-R^2 e^\mu$ and then simplifying, we obtain the next basic field equation

$$e^\mu (2R\ddot{R} + \dot{R}^2 + 1) - R'^2 = -\kappa \frac{\tau}{c} R^2 e^\mu p_{eff}. \quad (50)$$

We restate the last basic field equation we need, which is (43),

$$2\dot{R}' - R'\dot{\mu} = 0. \quad (51)$$

Equations (48), (50), (51) correspond to Carmeli's[1] eqns. (4.3.31), (4.3.29) and (4.3.30) respectively.

4 Solutions to the Field Equations

Equation (51) can be partially integrated with respect to $x^0 = \tau v$, keeping r constant. Integrating it we have

$$\int_{R'_o}^{R'} 2 \frac{\partial(X')}{X'} = \int_{\mu_o}^{\mu} \partial Y, \quad (52)$$

with the result

$$2 \ln(R'/R'_o) = \mu - \mu_o, \quad (53)$$

which can be put in the form

$$\frac{(R')^2}{e^\mu} = \frac{(R'_o)^2}{e^{\mu_o}} = 1 + f, \quad (54)$$

where f is an arbitrary function. The integration constants R'_o and μ_o are both evaluated at some particular value of coordinate $x^0 = \tau v_o$ so they can only be functions of coordinate r . Thus f is a function of r only. We can then write (54) in the useful form

$$e^\mu = \frac{(R')^2}{1 + f(r)}. \quad (55)$$

Since the l.h.s. of (55) is positive definite, the r.h.s. is likewise, implying that the condition on $f(r)$ is

$$1 + f(r) > 0. \quad (56)$$

The solution (55) is the same as[1], eqn. (4.3.16).

For the function R we assume the general form

$$R = R_r R_v, \quad (57)$$

where

$$R_r = R_r(r) \quad (58)$$

is a function of r only and

$$R_v = R_v(\tau v) \quad (59)$$

is a function of coordinate τv only.

Before proceeding with the solution of (50), which has the pressure p_{eff} , we first determine the relevant partial derivatives of (57),

$$R' = R'_r R_v, \quad (60)$$

$$\dot{R} = R_r \dot{R}_v, \quad (61)$$

$$\ddot{R} = R_r \ddot{R}_v. \quad (62)$$

Substituting for e^μ and the above derivatives into (50) yields

$$\begin{aligned} & \left(\frac{(R'_r)^2 R_v^2}{1 + f(r)} \right) [2R_r^2 R_v \ddot{R}_v + R_r^2 \dot{R}_v^2 + 1] - (R'_r)^2 R_v^2 \\ &= -\kappa \frac{\tau}{c} \left(\frac{(R'_r)^2 R_v^2}{1 + f(r)} \right) R_r^2 R_v^2 p_{eff}. \end{aligned} \quad (63)$$

Multiply (63) by $(1 + f(r))/(R'_r)^2 R_v^2$, gather terms and simplify to obtain

$$2R_v \ddot{R}_v + (\dot{R}_v)^2 + \kappa \frac{\tau}{c} R_v^2 p_{eff} = \frac{f(r)}{R_r^2} = f_o. \quad (64)$$

Assuming the pressure p_{eff} is not a function of r , then since the l.h.s. of (64) is a function of $x^0 = \tau v$ only and the r.h.s. is a function of r only, they both must equal a constant f_o . From the r.h.s. we conclude that

$$f(r) = f_o R_r^2. \quad (65)$$

We next solve (48) which contains the mass density ρ_{eff} . However, we need to first determine the derivatives of the function μ . Using (55) we obtain

$$\mu' = \frac{2R_r''}{R_r} - \frac{f'}{1 + f}, \quad (66)$$

$$\dot{\mu} = 2 \frac{\dot{R}_v}{R_v}. \quad (67)$$

Substituting for the functions and derivatives in (48) we obtain in unsimplified form

$$\begin{aligned} & \left(\frac{1 + f}{(R'_r)^2 R_v^2} \right) \left[\left(\frac{R'_r R_v}{R_r R_v} \right) \left(\frac{2R_r''}{R_r R_v} - \frac{f'}{1 + f} \right) - \left(\frac{R'_r R_v}{R_r R_v} \right)^2 - \frac{2R_r'' R_v}{R_r R_v} \right] \\ & + \left(\frac{2\dot{R}_v}{R_v} \right) \left(\frac{R_r \dot{R}_v}{R_r R_v} \right) + \left(\frac{R_r \dot{R}_v}{R_r R_v} \right)^2 + \frac{1}{R_r^2 R_v^2} = \kappa \tau^2 \rho_{eff}. \end{aligned} \quad (68)$$

This simplifies to

$$3(\dot{R}_v)^2 - \kappa\tau^2 \rho_{eff} R_v^2 = - \left(\frac{1+f}{(R'_r)^2} \right) \left[\frac{R'_r}{R_r} \left(\frac{2R''_r}{R'_r R_v} - \frac{f'}{1+f} \right) - \left(\frac{R'_r}{R_r} \right)^2 - \frac{2R''_r}{R_r} \right] - \frac{1}{R_r^2} = \mathcal{F}(v, r), \quad (69)$$

where in general \mathcal{F} is a function of v and r .

For the function R_r we now assume the simple form

$$R_r = r. \quad (70)$$

Then we have for its derivatives,

$$R'_r = 1, \quad (71)$$

$$R''_r = 0. \quad (72)$$

As we can see, $R''_r = 0$ implies that its terms along with the $1/R_v$ factor drops out of (69).

Substituting the solution values (70), (71) and (72) into the r.h.s. of (69) and simplifying we obtain a first order differential equation of the function f given by

$$rf' + f - \mathcal{F}r^2 = rf' + f - F_o r^2 = 0, \quad (73)$$

where $\mathcal{F}(v, r) = F_o$ because the r.h.s. of (69) has become a function of r only, while the l.h.s. is a function of v only, so they both must equal the constant F_o . By (65) and (70) we obtain

$$f(r) = f_o R_r^2 = f_o r^2, \quad (74)$$

which implies that, for this solution of $R_r = r$,

$$F_o = 3f_o. \quad (75)$$

If $f_o \neq 0$ then (74) is the solution to the inhomogeneous differential equation (73). If $f_o = 0$ then the homogeneous solution is

$$f(r) = -\frac{2GM}{c^2 r}, \quad (76)$$

where the coordinate system is centered on the central mass M . The general solution to (73) is the sum of (74) and (76),

$$f(r) = f_o r^2 - \frac{2GM}{c^2 r}. \quad (77)$$

From (69) we have

$$(\dot{R}_v)^2 = \frac{1}{3}\kappa\tau^2 R_v^2 \rho_{eff} + f_o. \quad (78)$$

To obtain a value for f_o from (78), assuming $\dot{R}_v \neq 0$ and $R_v \neq 0$, multiply the l.h.s. by R_v^2/R_v^2 and rearrange the result to obtain

$$\begin{aligned} f_o &= \frac{1}{c^2} H^2 R_v^2 - \frac{1}{3} \kappa \tau^2 R_v^2 \rho_{eff} \\ &= \frac{H^2 R_v^2}{c^2} \left(1 - \frac{8\pi G}{3H^2} \rho_{eff} \right) \\ &= \frac{H^2 R_v^2}{c^2} \left(1 - \frac{\rho_{eff}}{\rho_c} \right), \end{aligned} \quad (79)$$

where H is the Hubble parameter¹ defined by

$$H \equiv -c \frac{\dot{R}_v}{R_v}, \quad (80)$$

and ρ_c is the critical density defined by

$$\rho_c \equiv \frac{3H^2}{8\pi G}. \quad (81)$$

Notice that (79) is true at any epoch of coordinate x^0 , so we will evaluate it at $x^0 = \tau v = 0$. We define the scale factor R_v , the Hubble parameter H , the effective mass density ρ_{eff} and the critical density ρ_c to have the values at the present epoch,

$$R_v(0) = 1, \quad (82)$$

$$H(0) = h = 1/\tau, \quad (83)$$

$$\rho_{eff}(0) = \rho_m + \rho_\Lambda, \quad (84)$$

$$\rho_c(0) = \rho_c, \quad (85)$$

where ρ_m is the mass density and ρ_c is the critical density at the present epoch $v = 0$. With the values from (82)-(85) put into (79) we have

$$\begin{aligned} f_o &= \frac{h^2}{c^2} \left[1 - \frac{(\rho_m + \rho_\Lambda)}{\rho_c} \right] \\ &= \frac{-1}{c^2 \tau^2} [\Omega_m + \Omega_\Lambda - 1] \\ &= -K, \end{aligned} \quad (86)$$

¹ To define H as a positive valued parameter in (80) a negative sign is attached to compensate for the property that as the expansion velocity v increases from the local frame the scale factor R_v decreases toward the origin of the big-bang. This is contrary to the FLRW definition because there as cosmic time t increases from the big-bang to the present the scale factor $a(t)$ also increases.

where

$$\Omega_m = \frac{\rho_m}{\rho_c}, \quad (87)$$

$$\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c}, \quad (88)$$

$$\rho_c = \frac{3h^2}{8\pi G} = \frac{3}{8\pi G\tau^2}, \quad (89)$$

$$K = (\Omega_m + \Omega_\Lambda - 1)/c^2\tau^2, \quad (90)$$

where Ω_m is the matter mass density parameter at the present epoch ($v = 0$), Ω_Λ is the constant vacuum mass density parameter, and K is the curvature. Note that the curvature K has the dimension of $[\text{length}]^{-2}$. As will be shown, the type of spatial geometry, hyperbolic (open), Euclidean (flat) or spherical (closed), is determined by the curvature K which in turn depends on the mass and vacuum densities. This compares with the standard model where the type of geometry is determined by the dimensionless curvature parameter $k = -1, 0$ or 1 , for open, flat or closed, respectively.

The scale radius R_0 is given by

$$R_0 = 1/\sqrt{|K|}. \quad (91)$$

With the value for f_o from (86), with no central mass M , e^μ takes the form

$$e^\mu = \frac{R_v^2}{1 - Kr^2}. \quad (92)$$

5 Cosmological Redshift

For $ds = 0$, (19) describes the isotropic expansion of the universe. Using (92) for e^μ , the expansion can be described by

$$\frac{dr}{dv} = \tau e^{-\mu/2} = \frac{\tau}{R_v} \sqrt{1 - (\Omega_m + \Omega_\Lambda - 1)(r^2/c^2\tau^2)}. \quad (93)$$

$$\frac{\tau dv}{R_v} = \frac{dr}{\sqrt{1 - Kr^2}}. \quad (94)$$

To show the relation of the scale factor R_v to the cosmological redshift z , suppose an observer in a galaxy A measures the expansion velocities of two other galaxies “g” and “g+ δ g” which is nearby galaxy “g”. The velocities for the galaxies are v_A and $v_A + \delta v_A$ respectively. Suppose that another observer in another galaxy O measures expansion velocities for galaxies “g” and “g+ δ g” and obtains the values v_O and $v_O + \delta v_O$ respectively. We assume that δv_A and δv_O are small compared to v_A and v_O , respectively, since galaxies “g” and “g+ δ g” are near each other. What can we say about the relationship between these measured velocities from galaxies A and O?

Assume the distance to galaxy A is r_A and the distance to galaxy O is r_O . The galaxies are comoving which implies that the distance r_{OA} between them is

constant. For the galaxy “g”, we integrate (94) between the points (v_A, r_A) and (v_O, r_O)

$$\int_{v_A}^{v_O} \frac{\tau dv}{R_v} = \int_{r_A}^{r_O} \frac{dr}{\sqrt{1 - Kr^2}}. \quad (95)$$

For the galaxy “g+ δg ” we integrate (94) between the points $(v_A + \delta v_A, r_A)$ and $(v_O + \delta v_O, r_O)$,

$$\int_{v_A + \delta v_A}^{v_O + \delta v_O} \frac{\tau dv}{R_v} = \int_{r_A}^{r_O} \frac{dr}{\sqrt{1 - Kr^2}}. \quad (96)$$

Subtracting (95) from (96) we obtain

$$\begin{aligned} \int_{v_A + \delta v_A}^{v_O + \delta v_O} \frac{\tau dv}{R_v} - \int_{v_A}^{v_O} \frac{\tau dv}{R_v} = \\ \int_{r_A}^{r_O} \frac{dr}{\sqrt{1 - Kr^2}} - \int_{r_A}^{r_O} \frac{dr}{\sqrt{1 - Kr^2}} = 0. \end{aligned} \quad (97)$$

After some manipulations of the integrals (97) reduces to

$$\frac{\tau \delta v_O}{R_v(v_O)} - \frac{\tau \delta v_A}{R_v(v_A)} = 0, \quad (98)$$

where $R_v(v_O)$ is the approximate value of the assumed slowly varying scale factor over the small velocity interval δv_O , and likewise for $R_v(v_A)$ over the small interval δv_A . (98) can be put into the form

$$\frac{\tau \delta v_O}{\tau \delta v_A} = \frac{R_v(v_O)}{R_v(v_A)}. \quad (99)$$

If in galaxy A the distance $\tau \delta v_A$ is determined by the measurement of λ_A of the wavelength of photons from galaxy “g + δg ”, then in galaxy O the corresponding measurement of the photons from the same galaxy will have a wavelength λ_O . The ratio of the two wavelengths is assumed to be given by (99),

$$\frac{\tau \delta v_O}{\tau \delta v_A} = \frac{R_v(v_O)}{R_v(v_A)} = \frac{\lambda_O}{\lambda_A} = 1 + z, \quad (100)$$

where z is the cosmological redshift of the photon. If we take galaxy O to be the local galaxy then $v_O = 0$ and we set the scale factor of the local galaxy to unity, $R_v(v_O) = 1$. Then setting $v_A = v$ for the velocity of a general galaxy A, (100) can be put into the familiar form of the scale factor redshift relation,

$$R_v(v) = \frac{1}{1 + z}, \quad (101)$$

where

$$1 + z = \frac{\lambda_O}{\lambda_A}, \quad (102)$$

where λ_A is the photon wavelength detected by distant galaxy A and λ_O is the wavelength observed in the local galaxy.

6 Mass Continuity

By substituting the r.h.s. of (78) into its relevant term in (64) and simplifying we obtain

$$2\ddot{R}_v = - \left(\frac{1}{3} \kappa \tau^2 \rho_{eff} + \kappa \frac{\tau}{c} p_{eff} \right) R_v. \quad (103)$$

We can obtain another expression for $2\ddot{R}_v$ by differentiating (78) with respect to (w.r.t.) x^0 which, after simplifying gives

$$2\ddot{R}_v = \frac{1}{3} \kappa \tau^2 \left(\frac{R_v^2}{\dot{R}_v} \right) \dot{\rho}_{eff} + \frac{2}{3} \kappa \tau^2 R_v \rho_{eff}. \quad (104)$$

Combining (103) and (104) and simplifying gives us the effective mass density continuity equation

$$\dot{\rho}_{eff} = -3 \left(\rho_{eff} + \frac{p_{eff}}{c\tau} \right) \left(\frac{\dot{R}_v}{R_v} \right). \quad (105)$$

Simplifying (105) yields

$$\dot{\rho} = -3 \left(\rho + \frac{p}{c\tau} \right) \left(\frac{\dot{R}_v}{R_v} \right). \quad (106)$$

This mass density continuity equation (106) is identical in form to the energy density continuity equation of the standard FLRW model except that here the rate of mass density change is w.r.t. expansion velocity v while the standard model rate is w.r.t. cosmic time t .

We choose an evolving equation of state parameter [8] w_e such that the pressure p is related to the mass density ρ by

$$p = w_e c\tau \rho, \quad (107)$$

where

$$w_e(R_v) = w_0 + (1 - R_v) w_a, \quad (108)$$

where w_0 and w_a are constants. The second term on the r.h.s. of (108) represents the evolution of the equation of state as a function of expansion velocity. In particular, this functional form allows for a state equation to vary from low dark energy influence when the scale factor R_v was small into becoming dominated by dark energy at the current unity scale factor. Substituting for p from (107) into (106) we obtain

$$\dot{\rho} = -3(1 + w_e) \rho \left(\frac{\dot{R}_v}{R_v} \right). \quad (109)$$

(109) can be put into the form

$$\frac{d\rho}{\rho} = -3(1 + w_0 + (1 - R_v) w_a) \frac{dR_v}{R_v}, \quad (110)$$

which upon integration yields

$$\rho = \rho_m e^{-3w_a(1-R_v)} R_v^{-3(1+w_0+w_a)}. \quad (111)$$

Divide (111) by ρ_c to get the mass density parameter

$$\Omega = \Omega_m e^{-3w_a(1-R_v)} R_v^{-3(1+w_0+w_a)}. \quad (112)$$

Converting R_v in terms of the redshift z from (101), (112) for $\Omega = \Omega(z)$ becomes

$$\Omega(z) = \Omega_m e^{-3w_a z/(1+z)} (1+z)^{3(1+w_0+w_a)}. \quad (113)$$

7 Acceleration of the Scale Factor

With the equation of state (107) the effective pressure becomes

$$p_{eff} = p - c\tau\rho_\Lambda = c\tau(w_e\rho - \rho_\Lambda). \quad (114)$$

By substituting the r.h.s. of (78) into its relevant term in (64) and simplifying we obtain

$$2\ddot{R}_v = -\left(\frac{1}{3}\kappa\tau^2\rho_{eff} + \kappa\frac{\tau}{c}p_{eff}\right)R_v. \quad (115)$$

Substitute for effective pressure from (114) into (115) and simplify to obtain the scale factor acceleration equation

$$2\ddot{R}_v = -\frac{\kappa\tau^2}{3}[(1+3w_e)\rho - 2\rho_\Lambda]R_v. \quad (116)$$

This can be put into the form

$$2c^2\tau^2\ddot{R}_v = -(1+3w_e)\Omega R_v + 2\Omega_\Lambda R_v, \quad (117)$$

where $\Omega = \Omega(v)$ is given by (112) and Ω_Λ is the vacuum mass density parameter. (117) exhibits a range of possible scenarios for accelerating and decelerating expansions depending on w_e and on the values for Ω_m and Ω_Λ .

It was discovered experimentally that the expanding universe makes a transition from accelerating to decelerating[10,11,12]. Assume that the transition took place at velocity v_t corresponding to a redshift z_t . Then, taking (117) and setting $\ddot{R}_v = 0$ at $R_v(v_t) = (1+z_t)^{-1}$ we have an expression for w_e in terms of the transition redshift z_t ,

$$[1+3w_0+3w_az_t(1+z_t)^{-1}]\Omega_{z_t} - 2\Omega_\Lambda = 0, \quad (118)$$

where $\Omega_{z_t} = \Omega(z_t)$. This expression can be used to obtain a value for z_t in terms of the fitted parameters Ω_m , Ω_Λ , w_0 and w_a . We will encounter the transition redshift in the section on modeling.

8 Entropy of the Black Hole Universe

The scale radius of the universe given by (91) can be put into the form

$$R_0 = c\tau / \sqrt{|\Omega_m + \Omega_\Lambda - 1|}. \quad (119)$$

The value of Ω_m is the mass density at the present epoch of cosmic time $t = 0$, recalling that in CGR the cosmic time is measured from the present time $t = 0$ increasing toward the big bang time $t = \tau$. Assuming the universe is a black hole[5] of radius R_0 , then the event horizon surface area A is given by

$$A = 4\pi R_0^2 = \frac{4\pi c^2 \tau^2}{|\Omega_m + \Omega_\Lambda - 1|}. \quad (120)$$

Then the Bekenstein-Hawking[6,7] entropy S of the black hole universe is given by

$$S = \frac{kc^3 A}{4\hbar G} = \frac{\pi k c^5 \tau^2}{\hbar G |\Omega_m + \Omega_\Lambda - 1|}, \quad (121)$$

where k is Boltzmann's constant and \hbar is Planck's constant over 2π . Multiplying the r.h.s. of (121) by ρ_c/ρ_c and simplifying we obtain

$$\rho_\Lambda = \frac{\rho_P}{S/k} + \rho_c - \rho_m, \quad (122)$$

where $\rho_\Lambda = \rho_c \Omega_\Lambda$, $\rho_m = \rho_c \Omega_m$ and ρ_P is the cosmological Planck mass density defined by

$$\rho_P = \frac{\pm \mathcal{M}_P}{\mathcal{L}_P^3} = \frac{\pm 3c^5}{8\hbar G^2}, \quad (123)$$

where

$$\mathcal{M}_P = \sqrt{\sqrt{\frac{3}{8}} \frac{\hbar c}{G}} \quad (124)$$

is the cosmological Planck mass and

$$\mathcal{L}_P = \hbar / \mathcal{M}_P c, \quad (125)$$

is the cosmological Planck length. By Eqs. (121-125), since $\rho_c - \rho_m > 0$ observationally, and since the entropy is always non-negative, then a positive vacuum mass density implies a positive cosmological Planck mass density and visa-versa for a negative vacuum density. A deeper analysis into the relation between black hole entropy and the vacuum mass density is beyond the scope of this paper.

9 Expansion of Universe

(78) can be put into the form

$$\begin{aligned}\dot{R}_v &= \frac{dR_v}{\tau dv} = \frac{-1}{c\tau} \sqrt{\Omega_{eff} R_v^2 - K c^2 \tau^2} \\ &= \frac{-1}{c\tau} \sqrt{(\Omega + \Omega_\Lambda) R_v^2 - \Omega_K},\end{aligned}\quad (126)$$

where we used (27) for Ω_{eff} , $\Omega = \Omega(v)$ is given by (112) and

$$\Omega_K = K c^2 \tau^2 = (\Omega_m + \Omega_\Lambda - 1) \quad (127)$$

is the curvature density parameter. We select the minus sign in the r.h.s. of (126) because R_v is assumed to be a decreasing function of v .

(19) describes the isotropic expansion of the universe. Using (92) for e^μ , the expansion can be described by

$$\frac{\tau dv}{R_v} = \frac{dr}{\sqrt{1 - K r^2}}. \quad (128)$$

From (126) we obtain

$$\tau dv = \frac{dR_v}{\dot{R}_v} = \frac{-c\tau dR_v}{\sqrt{(\Omega + \Omega_\Lambda) R_v^2 - \Omega_K}}. \quad (129)$$

Substituting for τdv from (129) into the l.h.s. of (128) and simplifying we derive

$$\frac{-c\tau dR_v}{\sqrt{(\Omega + \Omega_\Lambda) R_v^4 - \Omega_K R_v^2}} = \frac{dr}{\sqrt{1 - K r^2}}. \quad (130)$$

From (101) the scale factor R_v is related to the cosmological redshift z by

$$R_v = (1 + z)^{-1}. \quad (131)$$

Differentiating (131) w.r.t. z gives

$$dR_v = -(1 + z)^{-2} dz. \quad (132)$$

Substituting for R_v and dR_v from (131) and (132) into (130) and simplifying yields the differential for the comoving distance relation

$$\frac{c dz}{\sqrt{\Omega + \Omega_\Lambda - \Omega_K (1 + z)^2}} = \frac{dr}{\sqrt{1 - K r^2}}, \quad (133)$$

where $\Omega = \Omega(z)$ is given by (113). The spatial geometry defined by the r.h.s. of (133) is either hyperbolic (open), Euclidean (flat) or spherical (closed) depending on curvature $K < 0$, $K = 0$ or $K > 0$, respectively. Since K is dependent on the mass and vacuum densities then the geometry of the universe is determined by Ω_m and Ω_Λ .

The expansion defined by (128) when substituted for the l.h.s. of (133) and combined with (131) yields the differential equation for the expansion velocity

$$dv = \frac{c dz}{\sqrt{(\Omega + \Omega_\Lambda) (1 + z)^2 - \Omega_K (1 + z)^4}}. \quad (134)$$

Integrating (134) we get for the expansion velocity as a function of redshift

$$v(z) = \int_0^z \frac{c dz'}{\sqrt{(\Omega + \Omega_A)(1+z')^2 - \Omega_K(1+z')^4}}. \quad (135)$$

10 Distances

The Hubble parameter defined by (80), using (126), is expressed in terms of velocity v ,

$$H(v) = -c \frac{\dot{R}_v}{R_v} = h \sqrt{\Omega + \Omega_A - \Omega_K R_v^{-2}}, \quad (136)$$

where $h = 1/\tau$, $\Omega = \Omega(v)$. In terms of redshift, using (131) for R_v ,

$$H(z) = h \sqrt{\Omega + \Omega_A - \Omega_K (1+z)^2}, \quad (137)$$

where $\Omega = \Omega(z)$ is defined by (113).

10.1 Comoving Distance

The comoving distance D_C is the integral of (133) and can be written in terms of the Hubble parameter using (137),

$$D_C = c\tau \int_0^z \frac{h dz'}{H(z')} = \int_0^r \frac{dr'}{\sqrt{1 - \Omega_K r'^2 / c^2 \tau^2}}. \quad (138)$$

10.2 Transverse Comoving Distance

The transverse comoving distance D_M is the coordinate distance r which is obtained from the inversion of (138), i.e., $D_M = r$. It takes the form

$$D_M = \frac{c\tau}{\sqrt{-\Omega_K}} \sinh \left(\frac{\sqrt{-\Omega_K}}{c\tau} D_C \right) \quad \text{for } \Omega_K < 0, \quad (139)$$

$$D_M = c\tau D_C \quad \text{for } \Omega_K = 0 \quad \text{and}$$

$$D_M = \frac{c\tau}{\sqrt{\Omega_K}} \sin \left(\frac{\sqrt{\Omega_K}}{c\tau} D_C \right) \quad \text{for } \Omega_K > 0.$$

10.3 Angular Diameter Distance

A physical source of size ΔS subtends an observed angle of $\Delta\theta$ on the sky given by the relation

$$\Delta\theta = \frac{\Delta S}{R}, \quad (140)$$

where $R = R_v r$ is the proper distance from the coordinate system origin to the source. Since $R_v = 1/(1+z)$ is the scale factor and $r = D_M$ is the coordinate distance, we define the angular diameter distance,

$$D_A = R_v r = \frac{D_M}{1+z}. \quad (141)$$

10.4 Luminosity Distance

It can be shown² that the source luminosity L transforms due to the universe expansion as $1/(1+z)^4$. Then the flux S from a source of luminosity L at proper distance $R = R_v r$ is given by

$$S = \left(\frac{L}{(1+z)^4} \right) \left(\frac{1}{4\pi R_v^2 r^2} \right) = \frac{L}{4\pi (1+z)^2 r^2}, \quad (142)$$

where we used $R_v = 1/(1+z)$. The bolometric flux S can also be defined for a source of luminosity L at distance D_L by

$$S = \frac{L}{4\pi D_L^2}. \quad (143)$$

Eliminating S between (142) and (143) we have the luminosity distance in terms of the redshift

$$D_L(z) = (1+z) \frac{c\tau}{\sqrt{-\Omega_K}} \sinh \left(\sqrt{-\Omega_K} \int_0^z \frac{hdz'}{H(z')} \right) \quad \text{for } \Omega_K < 0, \quad (144)$$

$$D_L(z) = (1+z) c\tau \int_0^z \frac{hdz'}{H(z')} \quad \text{for } \Omega_K = 0 \quad \text{and}$$

$$D_L(z) = (1+z) \frac{c\tau}{\sqrt{\Omega_K}} \sin \left(\sqrt{\Omega_K} \int_0^z \frac{hdz'}{H(z')} \right) \quad \text{for } \Omega_K > 0,$$

where we also used the l.h.s. of (138), (139) and the Hubble parameter defined by (137),

$$H(z) = h \sqrt{\Omega + \Omega_\Lambda - \Omega_K (1+z)^2}, \quad (145)$$

where $\Omega = \Omega(z)$ from (113),

$$\Omega(z) = \Omega_m e^{-3w_a z/(1+z)} (1+z)^{3(1+w_0+w_a)}. \quad (146)$$

² Essentially, due to the expansion, the radiation density decreases by a factor $1/(1+z)^3$ due to the increase in volume and each photon energy decreases by a factor $1/(1+z)$ due to the cosmological redshift.

11 Model Applications

We give a view of the cosmology by applying it to a small combined set[12, 13] total of 157 high redshift SNe Ia data, distance moduli and errors ($\mu_B \pm \sigma_B$) but not systematic errors. Since we are expecting a scale factor transition from accelerated to decelerated expansion at low redshift we require that at the origin the scale factor acceleration $\ddot{R}_v(z=0) > 0$. The standard distance modulus relation $m(z)$ is given by

$$m(z) - M_B = 5 \log(D_L(z)) + 25 + a, \quad (147)$$

where M_B is the absolute magnitude of a standard supernova at the peak of its light-curve, $D_L(z)$ is the luminosity distance (144) and a is an arbitrary zero point offset[16]. For all our examples, the set of parameters are pre-selected by trial and error and then a final fit is made of the distance modulus relation to the data varying only the offset parameter a .

For the first example we take for the mass density parameter

$$\Omega_m = \Omega_b/h_0^2, \quad (148)$$

where Ω_b is the baryon density parameter, and $h_0 = h/100 \text{ km/s/Mpc}$. From (11) this gives $h_0 = 0.7217$. We use a value of $\Omega_b = 0.020$ from[14]. Therefore, our value for the mass density parameter is

$$\Omega_m = 0.038. \quad (149)$$

By a few trials we found a good fit for a value of the vacuum density parameter

$$\Omega_\Lambda = -0.019. \quad (150)$$

This defines an open universe since the curvature $K = (\Omega_m + \Omega_\Lambda - 1)/c^2\tau^2 = -0.981/c^2\tau^2 < 0$. We consider an evolving dark energy state with $w_0 = -1.0$ and $w_a = +1.0$.

Fig. 1 shows that the acceleration \ddot{R}_v starts out positive at $z = 0$ and transitions to negative around $z = 0.84$. The detail at the acceleration transition is displayed by Fig. 2. In Fig. 3 is the plot of the distance moduli for the SNe Ia data. The theoretical distance modulus (147) is shown by the solid line. We obtained a fitted value $a = 0.164$ with $\chi^2/157 = 8.809$.

Figs. 4-7 display the various distance relations all with the same parameters from this first example. Distances are in units of $c\tau = 4158 \text{ Mpc}$.

For the second example we select no dark matter

$$\Omega_m = 0.038, \quad (151)$$

with positive vacuum density

$$\Omega_\Lambda = 0.4. \quad (152)$$

This again defines an open universe since $K = -0.562/c^2\tau^2 < 0$. The equation of state parameters are $w_0 = -1$ and $w_a = +4.3$.

For the third example we select baryonic and dark matter

$$\Omega_m = 0.3, \quad (153)$$

Table 1 Curve fit parameters.

Ex.	Ω_m	Ω_Λ	$Kc\tau$	w_0	w_a	a	$\chi^2/157$	z_t
1	0.038	-0.019	-0.981	-1.0	+1.0	+0.164	8.809	0.84
2	0.038	+0.4	-0.562	-1.0	+4.3	+0.064	8.243	0.78
3	0.3	+0.7	0	-1.0	+4.3	-0.043	8.626	0.46

with positive vacuum density

$$\Omega_\Lambda = 0.7, \quad (154)$$

where $\Omega_m + \Omega_\Lambda = 1$ which gives a flat space curvature $K = 0$ as in the standard FLRW Lambda-Cold-Dark-Matter (LCDM) model. We use the same equation of state $w_0 = -1$ and $w_a = +4.3$ as the second example. The results of these two examples are combined in the plots.

Fig. 8 shows details of the acceleration \ddot{R}_v . The thick curve is for $\Omega_m = 0.038$, $\Omega_\Lambda = 0.4$, $w_0 = -1.0$ and $w_a = 4.3$ with transition $z_t \approx 0.78$. The thin curve is for $\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$, $w_0 = -1.0$ and $w_a = 4.3$ with transition $z_t \approx 0.46$ as was reported[11,12]. (We actually varied w_a to get a smaller χ^2 and in the process honed in on $z_t = 0.46$.) Fig. 9 shows the plots of the distance moduli for the SNe Ia data fitted to these two sets of parameters. The thick (upper) curve is for $\Omega_m = 0.038$, $\Omega_\Lambda = 0.4$, $w_0 = -1$, $w_a = 4.3$ and fitted offset $a = 0.064$ with $\chi^2/157 = 8.243$. The thin (lower) curve is for $\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$, $w_0 = -1$, $w_a = 4.3$ and fitted offset $a = -0.043$ with $\chi^2/157 = 8.626$. The examples and parameters are summarized in Table 1.

For a brief summary, Example (Ex.) 2 has the smallest $\chi^2/N = 8.243$ where $N = 157$, while Ex. 1 has the largest $\chi^2/N = 8.809$, with Ex. 3 in between with a $\chi^2/N = 8.626$. However, both Ex.2 and Ex. 3 have exceptionally high evolution ($w_0 = -1.0$, $w_a = +4.3$) toward higher redshift, of what may be termed a “anti-phantom” energy, which might be characterized as physically unlikely because it has not been observed so far. On the other hand, Ex. 1 has an evolution ($w_0 = -1.0$, $w_a = +1.0$) of “anti-dark” energy toward higher redshift which cancels the dark energy development, which is physically more plausible since it involves dark energy for which there is observable evidence. Thus, qualitatively, Ex. 1 is given a “best fit” rating overall. Ref. (Riess, et al.[11], Sect. 4.3) for a nice exposition on dark energy.

These three examples provide a glimpse at the cosmology. Consider that there are $N_C = 2 \times 3^3 = 54$ possible categories of the parameter space: Ω_m with no dark matter or dark matter, Ω_Λ , w_0 and w_a each either negative, zero or positive. Obviously, systematic analyses and larger data sets are required to narrow down the parameter space.

12 Carmeli Cosmology

The 4-D space-velocity cosmology of Carmeli[1,2] is based on the premise of a constant scale factor R_v . Assume

$$\ddot{R}_v = \dot{R}_v = 0, \quad (155)$$

$$R_v = 1,$$

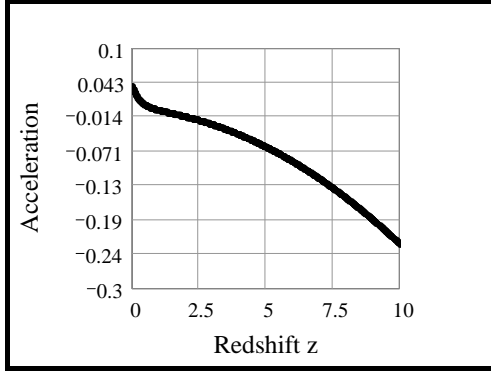


Fig. 1 Scale factor acceleration \ddot{R}_v . $\Omega_m = 0.038$, $\Omega_\Lambda = -0.019$, $w_0 = -1.0$ and $w_a = 1.0$.

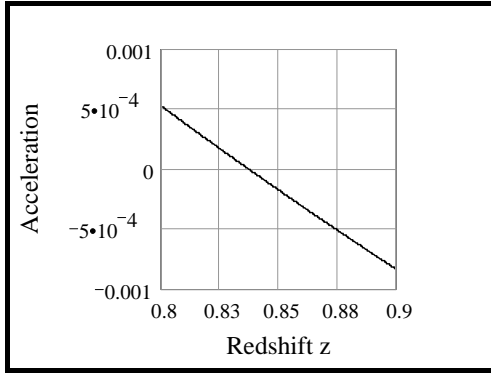


Fig. 2 Scale factor acceleration \ddot{R}_v at transition redshift $z_t \approx 0.84$. $\Omega_m = 0.038$, $\Omega_\Lambda = -0.019$, $w_0 = -1.0$ and $w_a = 1.0$.

with

$$R = R_r = r, \quad (156)$$

$$R' = R'_r = 1, \quad (157)$$

$$R'' = R''_r = 0. \quad (158)$$

The key equations are (64), (69) and (73). With the values Eqs. (156)-(158) substituted into (69) we get

$$-\kappa\tau^2\rho_{eff} = \frac{f'}{r} + \frac{f}{r^2} = F_o, \quad (159)$$

where F_o is a constant since ρ_{eff} is assumed to be a constant with respect to v and r . However, it is assumed that ρ_{eff} is a function of the cosmic time. For the inhomogeneous case where $F_o \neq 0$ the solution of (159) is

$$f(r) = f_o r^2, \quad (160)$$

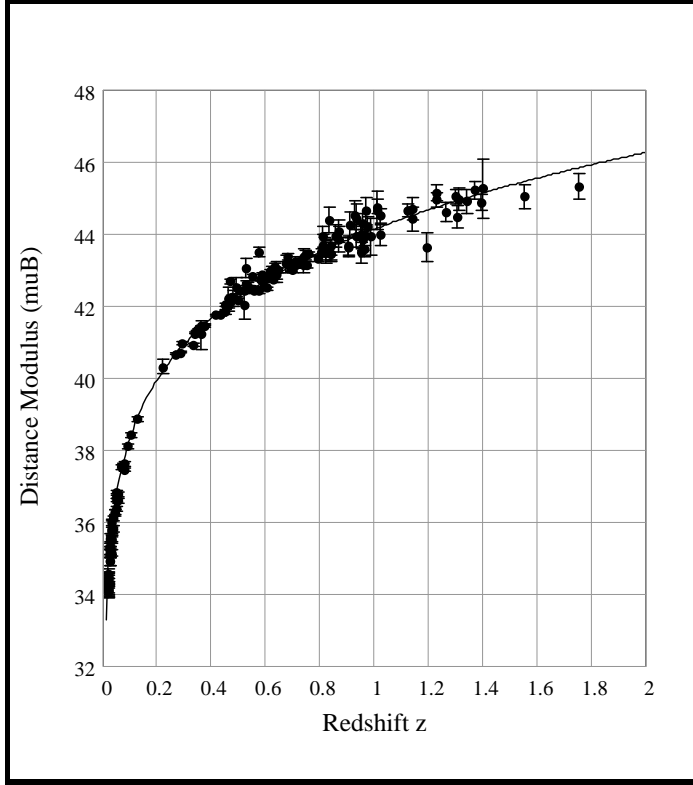


Fig. 3 SNe Ia distance moduli. $\Omega_m = 0.038$, $\Omega_\Lambda = -0.019$, $w_0 = -1.0$ and $w_a = 1.0$. Points are $(\mu_B \pm \sigma_B)$ from [12, 13]. Solid curve is $m(z) - M_B = 5 \log(c\tau D_L(z)) + 25 + a$ with fitted $a = 0.164$ and $\chi^2/157 = 8.809$.

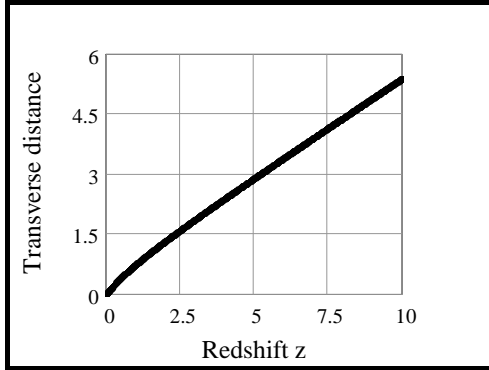


Fig. 4 Transverse comoving distance D_M . $\Omega_m = 0.038$, $\Omega_\Lambda = -0.019$, $w_0 = -1.0$ and $w_a = 1.0$.

where f_o is a constant. And, for the homogeneous case where $F_o = 0$ the solution is

$$f(r) = -\frac{2GM}{c^2 r}, \quad (161)$$

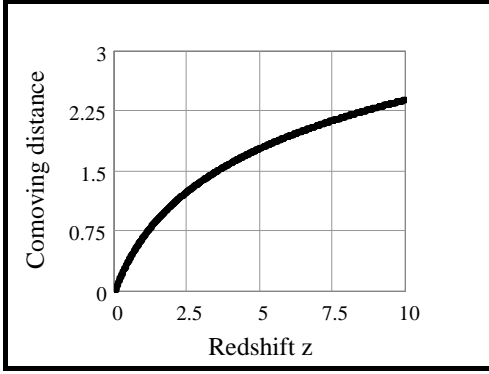


Fig. 5 Comoving distance D_C . $\Omega_m = 0.038$, $\Omega_\Lambda = -0.019$, $w_0 = -1.0$ and $w_a = 1.0$.

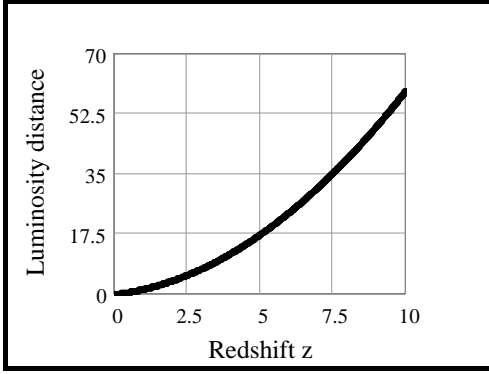


Fig. 6 Luminosity distance D_L . $\Omega_m = 0.038$, $\Omega_\Lambda = -0.019$, $w_0 = -1.0$ and $w_a = 1.0$.

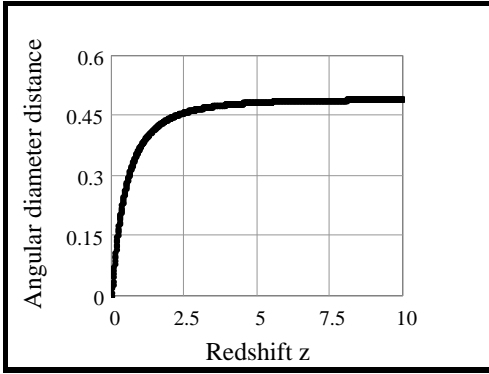


Fig. 7 Angular diameter distance D_A . $\Omega_m = 0.038$, $\Omega_\Lambda = -0.019$, $w_0 = -1.0$ and $w_a = 1.0$.

where M is a constant point mass centered at the origin of coordinates. The general solution is the sum of (160) and (161),

$$f(r) = f_o r^2 - \frac{2GM}{c^2 r}, \quad (162)$$

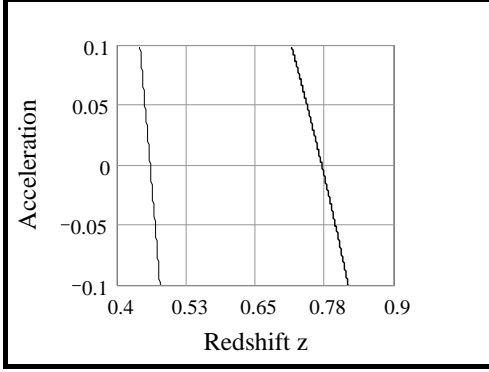


Fig. 8 Scale factor acceleration \ddot{R}_v at transition redshift. Thin (left) curve is for $\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$, $w_0 = -1.0$ and $w_a = 4.3$ with transition $z_t \approx 0.46$. Thick (right) curve is for $\Omega_m = 0.038$, $\Omega_\Lambda = 0.4$, $w_0 = -1.0$ and $w_a = 4.3$ with transition $z_t \approx 0.78$.

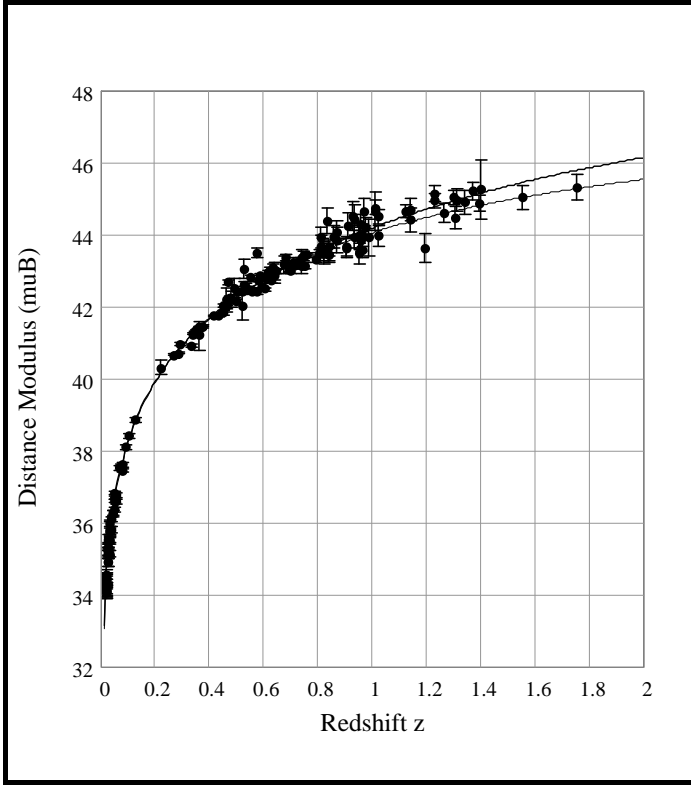


Fig. 9 SNe Ia distance moduli. Points are $(\mu_B \pm \sigma_B)$ from [12, 13]. The thick (upper) curve is for $\Omega_m = 0.038$, $\Omega_\Lambda = 0.4$, $w_0 = -1$, $w_a = 4.3$ and fitted offset $a = 0.064$ with $\chi^2/157 = 8.243$. The thin (lower) curve is for $\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$, $w_0 = -1$, $w_a = 4.3$ and offset $a = -0.043$ with $\chi^2/157 = 8.626$.

where the system of coordinates is centered on the central mass M . Using (162) in (159) gives the value

$$F_o = 3f_o. \quad (163)$$

From (64), with $\ddot{R}_v = \dot{R}_v = 0$ and $R_v = 1$ we get

$$\kappa \frac{\tau}{c} p_{eff} = \frac{f(r)}{r^2} = f_o. \quad (164)$$

Substituting for $f(r)$ from (162) into (164) gives the effective pressure

$$p_{eff} = \frac{c}{\kappa\tau} f_o - \frac{c\tau M}{4\pi r^3}. \quad (165)$$

The value of f_o is obtained from (78) with $\dot{R}_v = 0$ and $R_v = 1$, so that at $v = 0$ it gives us

$$f_o = -\frac{1}{3}\kappa\tau^2\rho_{eff}. \quad (166)$$

With this value for f_o we obtain from (165) the effective pressure

$$p_{eff} = -\frac{1}{3}c\tau(\rho_{eff} + \rho_M(r)), \quad (167)$$

where

$$\rho_M(r) = \frac{3M}{4\pi r^3} \quad (168)$$

is the central mass density, which at this point is a convenient mathematical construct. The function (55) for e^μ is now given by

$$e^\mu = \frac{1}{1+f(r)} = \frac{1}{1 - \frac{1}{3}\kappa\tau^2 r^2 \rho_{eff} - 2GM/c^2 r}, \quad (169)$$

where

$$1 - \frac{1}{3}\kappa\tau^2 r^2 \rho_{eff} - 2GM/c^2 r > 0. \quad (170)$$

We make note that Carmeli did not define an effective pressure, but simply used an ordinary pressure. In our analysis the effective pressure is used. We could setup an equation of state for Carmeli cosmology that is defined, taking the central mass $M = 0$ in (167),

$$p_{eff} = w c \tau \rho_{eff}, \quad (171)$$

where $w = -1/3$. Looking at (107) this implies that $w_0 = -1/3$ and $w_a = 0$.

In Carmeli cosmology the effective mass density $\rho_{eff} = \rho - \rho_c$, so we equate it with our definition (27) and obtain

$$\rho_{eff} = \rho_m + \rho_\Lambda = \rho - \rho_c, \quad (172)$$

from which, with $\rho_m = \rho$, we get

$$\rho_\Lambda = -\rho_c, \quad (173)$$

where $\rho_c = 3/8\pi G\tau^2$ is the critical density defined by (89). Using (26) this yields

$$\rho_\Lambda = -\frac{\Lambda}{\kappa\tau^2} = -\rho_c, \quad (174)$$

$$\Lambda = \kappa\tau^2\rho_c = \frac{3}{c^2\tau^2}. \quad (175)$$

Allowing for the c^2 this is the relation for Λ which was reported[15]. However, this does not fix the value of Λ but it appears to experimentally have the same order of magnitude[2]. Using (172) and (173), the effective mass density parameter Ω_{eff} is given by

$$\Omega_{eff} = \frac{\rho_m + \rho_\Lambda}{\rho_c} = \Omega_m + \Omega_\Lambda, \quad (176)$$

where

$$\Omega_m = \frac{\rho_m}{\rho_c} \quad (177)$$

and

$$\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c} = \frac{-\rho_c}{\rho_c} = -1. \quad (178)$$

With no central mass, $M = 0$, and using Eqs. (176)-(178) then (169) takes the form

$$e^\mu = \frac{1}{1 - (\Omega_{eff}) r^2 / c^2 \tau^2} = \frac{1}{1 + (1 - \Omega_m) r^2 / c^2 \tau^2}. \quad (179)$$

Equation (179) is written in that form because $\Omega_m < 1$ at the present epoch. The curvature $K = (\Omega_m - 1)/c^2 \tau^2 < 0$ which implies that the Carmeli universe has a hyperbolic (open) spatial geometry.

Setting $ds = 0$ in (19) for the expansion of the universe gives the differential equation

$$\tau dv = \frac{dr}{\sqrt{1 + (1 - \Omega_m) r^2 / c^2 \tau^2}}. \quad (180)$$

Upon integration, assuming $1 - \Omega_m > 0$, (180) yields for the expansion velocity

$$v = \frac{c \sinh^{-1}(\sqrt{1 - \Omega_m} r / c \tau)}{\sqrt{1 - \Omega_m}}. \quad (181)$$

Inverting (181) we obtain the velocity-distance relation

$$r = \frac{c \tau \sinh(\sqrt{1 - \Omega_m} v / c)}{\sqrt{1 - \Omega_m}}. \quad (182)$$

When the central mass $M > 0$, the expression (169) takes the form

$$e^\mu = \frac{1}{1 + (1 - \Omega_m) r^2 / c^2 \tau^2 - 2GM / c^2 r}. \quad (183)$$

Then the differential equation for the universe expansion (19) takes the form

$$\tau dv = \frac{dr}{\sqrt{1 + (1 - \Omega_m) r^2 / c^2 \tau^2 - 2GM / c^2 r}}. \quad (184)$$

The velocity-distance relation is then given by

$$v = h \int_0^r \frac{dr'}{\sqrt{1 + (1 - \Omega_m) r'^2 / c^2 \tau^2 - 2GM / c^2 r'}}, \quad (185)$$

where the expansion is centered on the point mass M . This model was recently described by Hartnett[18].

This is the basics of the cosmology. For $\Omega_m < 1$ the curvature $K = (\Omega_m - 1)/c^2\tau^2 < 0$ defines an open universe. Since the scale factor R_v is assumed to be constant, a velocity-redshift relation must be obtained by other methods. The special relativistic Doppler velocity-redshift relation

$$\frac{v}{c} = \frac{(1+z)^2 - 1}{(1+z)^2 + 1} \quad (186)$$

is often used. However, since here the scale factor R_v is constant, the redshift z is assumed to be a function of the cosmic time. Also, the evolution of the mass density parameter Ω_m is assumed to be in the time domain and its variation also must be obtained by other methods. A further limitation is that in (180), the restriction that $1 + (1 - \Omega_m)r^2/c^2\tau^2 > 0$ requires $\Omega_m \leq 2$ approximately. This makes it difficult for high redshift data analysis[16,17]. On the other hand, when cosmic time is added as a fifth dimension the CGR 5-D time-space-velocity model is applicable to galaxy dynamics because the cosmological redshift across a galaxy region is nearly constant[4,19,20,21].

13 Conclusion

A general solution to the Einstein field equations has been obtained for the four dimensional space-velocity Cosmological General Relativity theory of Carmeli. This development provides the tools necessary for the analysis of astrophysical data. In particular, a redshift-distance relation is given, analogous to the standard FLRW cosmology, and an evolving equation of state is provided. An analysis of high redshift SNe Ia data was made to show the efficacy of the cosmology. Model examples were given with only baryonic matter and dark energy in hyperbolic space and with baryonic plus dark matter and dark energy in flat space. Finally, Carmeli's cosmology in 4-D was obtained.

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